Lectures on Dynamic Systems and Control

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Chapter 26

Balanced Realization

26.1 Introduction

One popular approach for obtaining a minimal realization is known as *Balanced Realization*. In this approach, a new state-space description is obtained so that the reachability and observability gramians are diagonalized. This defines a new set of *invariant* parameters known as Hankel singular values. This approach plays a major role in model reduction which will be highlighted in this chapter.

26.2 Balanced Realization

Let us start with a system G with minimal realization

$$G \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

As we have seen in an earlier lecture, the controllability gramian P, and the observability gramian Q are obtained as solutions to the following Lyapunov equations

$$\begin{aligned} AP + PA' + BB' &= 0 \\ A'Q + QA + C'C &= 0. \end{aligned}$$

P and Q are symmetric and since the realization is minimal they are also positive definite. The eigenvalues of the product of the controllability and observability gramians play an important role in system theory and control. We define the Hankel singular values, σ_i, as the square roots of the eigenvalues of PQ

$$\sigma_i \stackrel{\triangle}{=} (\lambda_i(PQ))^{\frac{1}{2}}.$$

We would like to obtain coordinate transformation, T, that results in a realization for which the controllability and observability gramians are equal and diagonal. The diagonal entries of the transformed controllability and observability gramians will be the Hankel singular values. With the coordinate transformation T the new system realization is given by

$$G \sim \left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right] = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & D \end{array}\right],$$

and the Lyapunov equations in the new coordinates are given by

$$\hat{A}(T^{-1}PT'^{-1}) + (T^{-1}PT'^{-1})\hat{A}' + \hat{B}\hat{B}' = 0$$
$$\hat{A}'(T'QT) + (T'QT)\hat{A} + \hat{C}'\hat{C} = 0.$$

Therefore the controllability and observability gramian in the new coordinate system are given by

$$\begin{aligned} \hat{P} &= T^{-1}PT'^{-1} \\ \hat{Q} &= T'QT. \end{aligned}$$

We are looking for a transformation T such that

$$\hat{P} = \hat{Q} = \Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix}.$$

We have the relation

$$\begin{aligned} (T^{-1}PT^{-1'})(T'QT) &= \Sigma^2, \\ T^{-1}PQT &= \Sigma^2. \end{aligned} \tag{26.1}$$

Since $Q = Q'$ and is positive definite, we can factor it as $Q = R'R$, where R is an invertible matrix. We can write equation 26.1 as $T^{-1}PR'RT = \Sigma^2$, which is equivalent to

$$(RT)^{-1}RPR'(RT) = \Sigma^2. \tag{26.2}$$

Equation 26.2 means that RPR' is similar to Σ^2 and is positive definite. Therefore, there exists an orthogonal transformation U, $U'U = I$, such that

$$RPR' = U\Sigma^2U'. \tag{26.3}$$

By setting $(RT)^{-1}U\Sigma^{\frac{1}{2}} = I$, we arrive at a definition for T and T^{-1} as

$$\begin{aligned} T &= R^{-1}U\Sigma^{\frac{1}{2}} \\ T^{-1} &= \Sigma^{-\frac{1}{2}}U'R. \end{aligned}$$

With this transformation it follows that

$$\begin{aligned} \hat{P} &= (\Sigma^{-\frac{1}{2}}U'R)P(R'U\Sigma^{-\frac{1}{2}}) \\ &= (\Sigma^{-\frac{1}{2}}U')(U\Sigma^2U')(U\Sigma^{-\frac{1}{2}}) \\ &= \Sigma, \end{aligned}$$

and

$$\begin{aligned} \hat{Q} &= (R^{-1}U\Sigma^{\frac{1}{2}})'R'R(R^{-1}U\Sigma^{\frac{1}{2}}) \\ &= (\Sigma^{\frac{1}{2}}U')(R'^{-1}R'RR^{-1})(U\Sigma^{\frac{1}{2}}) \\ &= \Sigma. \end{aligned}$$

26.3 Model Reduction by Balanced Truncation

Suppose we start with a system

$$G \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

where A is asymptotically stable. Suppose T is the transformation that converts the above realization to a balanced realization, with

$$G \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & D \end{array}\right],$$

and $\hat{P} = \hat{Q} = \Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$. In many applications it may be beneficial to only consider the subsystem of G that corresponds to the Hankel singular values that are larger than a certain small constant. For that reason, suppose we partition Σ as

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}$$

where Σ_2 contains the small Hankel singular values. We can partition the realization of G accordingly as

$$G \sim \left[\begin{array}{cc|c} \hat{A}_{11} & \hat{A}_{12} & \hat{B}_1 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{C}_2 & D \end{array}\right].$$

Recall that the following Lyapunov equations hold

$$\hat{A}\Sigma + \Sigma\hat{A}' + \hat{B}\hat{B}' = 0$$

$$\hat{A}'\Sigma + \Sigma\hat{A} + \hat{C}'\hat{C} = 0,$$

which can be expanded as

$$\begin{bmatrix} \hat{A}_{11}\Sigma_1 & \hat{A}_{12}\Sigma_2 \\ \hat{A}_{21}\Sigma_1 & \hat{A}_{22}\Sigma_2 \end{bmatrix} + \begin{bmatrix} \Sigma_1\hat{A}'_{11} & \Sigma_1\hat{A}'_{21} \\ \Sigma_2\hat{A}'_{12} & \Sigma_2\hat{A}'_{22} \end{bmatrix} + \begin{bmatrix} \hat{B}_1\hat{B}'_1 & \hat{B}_1\hat{B}'_2 \\ \hat{B}_2\hat{B}'_1 & \hat{B}_2\hat{B}'_2 \end{bmatrix} = 0,$$

$$\begin{bmatrix} \hat{A}'_{11}\Sigma_1 & \hat{A}'_{21}\Sigma_2 \\ \hat{A}'_{12}\Sigma_1 & \hat{A}'_{22}\Sigma_2 \end{bmatrix} + \begin{bmatrix} \Sigma_1\hat{A}_{11} & \Sigma_1\hat{A}_{12} \\ \Sigma_2\hat{A}_{21} & \Sigma_2\hat{A}_{22} \end{bmatrix} + \begin{bmatrix} \hat{C}'_1\hat{C}_1 & \hat{C}'_1\hat{C}_2 \\ \hat{C}'_2\hat{C}_1 & \hat{C}'_2\hat{C}_2 \end{bmatrix} = 0.$$

From the above two matrix equations we get the following set of equations

$$\hat{A}_{11}\Sigma_1 + \Sigma_1\hat{A}'_{11} + \hat{B}_1\hat{B}'_1 = 0 \tag{26.4}$$

$$\hat{A}_{12}\Sigma_2 + \Sigma_1\hat{A}'_{21} + \hat{B}_1\hat{B}'_2 = 0 \tag{26.5}$$

$$\hat{A}_{22}\Sigma_2 + \Sigma_2\hat{A}'_{22} + \hat{B}_2\hat{B}'_2 = 0 \tag{26.6}$$

$$\hat{A}'_{11}\Sigma_1 + \Sigma_1\hat{A}_{11} + \hat{C}'_1\hat{C}_1 = 0 \tag{26.7}$$

$$\hat{A}_{21}'\Sigma_2 + \Sigma_1\hat{A}_{12} + \hat{C}_1'\hat{C}_2 = 0 \tag{26.8}$$

$$\hat{A}_{22}'\Sigma_2 + \Sigma_2\hat{A}_{22} + \hat{C}_2'\hat{C}_2 = 0. \tag{26.9}$$

From this decomposition we can extract two subsystems

$$G_1 \sim \left[\begin{array}{c|c} \hat{A}_{11} & \hat{B}_1 \\ \hline \hat{C}_1 & D \end{array}\right], G_2 \sim \left[\begin{array}{c|c} \hat{A}_{22} & \hat{B}_2 \\ \hline \hat{C}_2 & D \end{array}\right].$$

Theorem 26.1 *G is an asymptotically stable system. If Σ_1 and Σ_2 do not have any common diagonal elements then G_1 and G_2 are asymptotically stable.*

Proof: Let us show that the subsystem

$$G_1 \sim \left[\begin{array}{c|c} \hat{A}_{11} & \hat{B}_1 \\ \hline \hat{C}_1 & D \end{array}\right]$$

is asymptotically stable. Since $\hat{A}_{11}$ satisfies the Lyapunov equation

$$\hat{A}_{11}\Sigma_1 + \Sigma_1\hat{A}_{11}' + \hat{B}_1\hat{B}_1' = 0$$

then it immediately follows that all the eigenvalues of $\hat{A}_{11}$ must be in the closed left half of the complex plane; that is, $Re\lambda_i(\hat{A}_{11}) \leq 0$. In order to show asymptotic stability we must show that $\hat{A}_{11}$ has no purely imaginary eigenvalues.

Suppose $j\omega$ is an eigenvalue of $\hat{A}_{11}$, and let v be an eigenvector associated with $j\omega$; $(\hat{A}_{11} - j\omega I)v = 0$. Assume that the Kernel of $(\hat{A}_{11} - j\omega I)$ is one-dimensional. The general case where there may be several independent eigenvectors associated with $j\omega$ can be handled by a slight modification of the present argument.

Equation 26.7 can be written as

$$(\hat{A}_{11} - j\omega I)'\Sigma_1 + \Sigma_1(A_{11} - j\omega I) + \hat{C}_1'\hat{C}_1 = 0$$

By multiplying the above equation by v on the right and v' on the left we get

$$v'(\hat{A}_{11} - j\omega I)'\Sigma_1 v + v'\Sigma_1(A_{11} - j\omega I)v + v'\hat{C}_1'\hat{C}_1 v = 0$$

which implies that $(\hat{C}_1 v)'(\hat{C}_1 v) = 0$, and this in turn implies that

$$\hat{C}_1 v = 0. \tag{26.10}$$

Again from equation 26.7 we get

$$(\hat{A}_{11} - j\omega I)'\Sigma_1 v + \Sigma_1(A_{11} - j\omega I)v + \hat{C}_1'\hat{C}_1 v = 0,$$

which implies that

$$(\hat{A}_{11} - j\omega I)'\Sigma_1 v = 0. \tag{26.11}$$

Now we multiply equation 26.4 from the right by $\Sigma_1 v$ and from the left by $v'\Sigma_1$ to get

$$v'\Sigma_1(\hat{A}_{11} - j\omega I)\Sigma_1^2 v + v'\Sigma_1^2(A_{11} - j\omega I)'\Sigma_1 v + v'\Sigma_1 B_1 B_1'\Sigma_1 v = 0.$$

This implies that $v'\Sigma_1 B_1)(B_1'\Sigma_1 v) = 0$, and $B_1'\Sigma_1 v = 0$. By multiplying equation 26.4 on the right by $\Sigma_1 v$ we get

$$(\hat{A}_{11} - j\omega I)\Sigma_1^2 v + \Sigma_1(A_{11} - j\omega I)'\Sigma_1 v + \hat{B}_1\hat{B}_1'\Sigma_1 v = 0$$

and hence

$$(\hat{A}_{11} - j\omega I)\Sigma_1^2 v = 0. \tag{26.12}$$

Since that the kernel of $(\hat{A}_{11} - j\omega I)$ is one dimensional, and both v and $\Sigma_1^2 v$ are eigenvectors, it follows that $\Sigma_1^2 v = \hat{\sigma}^2 v$, where $\hat{\sigma}$ is one of the diagonal elements in Σ_1^2.

Now multiply equation 26.5 on the left by $v'\Sigma_1$ and equation 26.8 by v' on the left we get

$$v'\Sigma_1\hat{A}_{12}\Sigma_2 + v'\Sigma_1^2\hat{A}_{21}' = 0 \tag{26.13}$$

and

$$v'\hat{A}_{21}'\Sigma_2 + v'\Sigma_1\hat{A}_{12} = 0. \tag{26.14}$$

¿From equations 26.13 and 26.14 we get that

$$-v'\hat{A}_{21}'\Sigma_2^2 + \hat{\sigma}^2 v'\hat{A}_{21}' = 0,$$

which can be written as

$$(v'\hat{A}_{21}')\left[-\Sigma_2^2 + \hat{\sigma}^2 I\right] = 0.$$

Since by assumption Σ_2^2 and Σ_1^2 have no common eigenvalues, then $\hat{\sigma}^2 I$ and Σ_2 have no common eignevalues, and hence $A_{21}v = 0$. We have

$$\begin{aligned}(\hat{A}_{11} - j\omega I)v &= 0\\ \hat{A}_{21}v &= 0,\end{aligned}$$

which can be written as

$$\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}\begin{bmatrix} v \\ 0 \end{bmatrix} = j\omega\begin{bmatrix} v \\ 0 \end{bmatrix}.$$

This statement implies that $j\omega$ is an eigenvalue of $\hat{A}$, which contradicts the assumption of the theorem stating that G is asymptotically stable.